Initial and Post Buckling of Axially Compressed Orthotropic Cylindrical Shells

M. S. El Naschie*
University College, London, England

IN a previous study of the initial post buckling of axially compressed cylindrical shells, it was found that this structure exhibits an unstable point of bifurcation for the free ends and the semi-infinite simply supported case.[1] This result resembles, in principle, the stability performance of many shell structures.

In the present work, this initial post buckling approach is extended to give information about the large deflection of cylindrical shells which exhibits orthogonal anistropy. The analysis represents an extension of the initial post buckling approach pioneered by Koiter.[2,3] Finally, a simple engineering approach to the lower stability limit[4,5] is presented.

I. Energy Functional

Considering the symmetrical buckling configuration of an axially compressed orthotropic cylindrical shell, it is easily verified that the potential energy functional can be written in the following intrinsic form[6]

$$V=2\pi r\int_0^{\ell}\left\{\frac{1}{2}D_x\dot{\psi}^2+\frac{1}{2}k_x\epsilon_x^2+P(\cos\psi(1-\epsilon_x)-1)+\frac{k_y}{2r^2}\left(\int\sin\psi(1+\epsilon_x)dx\right)^2\right\}dx \tag{1}$$

where ψ is the angle of rotation, ϵ_x is the axial strain in the longitudinal direction, $\dot{()}=(d(\;)/dx)$, $0\leq x\leq\ell$, ℓ is the length of the cylinder, D_x is the bending stiffness, k_x is the axial stiffness in the longitudinal direction, k_y is the axial stiffness in the circumferential direction, and P is the axial pressure.

Setting ϵ_x equal to zero,[1] i.e., $k_x=\infty$, noting that the convected transformation $V\rightarrow W$ renders trivial,[1] where W is Thompson's transformed energy functional[1,6,7] which is related to a set of sliding coordinate systems and expanding W as

$$W=W_1+W_2+W_3+W_4+W_5+W_6+\ldots \tag{2}$$

we obtain

$$W_1=W_3=W_5=\ldots=0 \tag{3}$$

$$W_2=\int_0^{\ell}\left[\frac{D_x}{2}\dot{\psi}^2-P\frac{\psi^2}{2}+\frac{k_y}{2r^2}\left(\int\psi dx\right)^2\right]2\pi r dx \tag{4}$$

$$W_4=\int_0^{\ell}\left[\frac{P}{24}\psi^4-k_y\left(\int\frac{\psi}{r^2}dx\cdot\int\frac{\psi^3}{3!}dx\right)\right]2\pi r dx \tag{5}$$

and

$$W_6=\int_0^{\ell}\left[-\frac{P}{6!}\psi^6+k_y\left(\int\frac{\psi}{r^2}dx\cdot\int\frac{\psi^5}{5!}dx\right)\right]2\pi r dx \tag{6}$$

We note that while it was sufficient to retain up to the fourth-order energy terms,[1] for the analysis of the initial post buckling, the present extended approach is taken up to the energy terms of the sixth order (W_6).

II. Initial Post Buckling Analysis

Expanding ψ as

$$\psi=\sum_i^{\infty}a_i\cos\frac{i\pi}{\ell}x \tag{7}$$

and regarding the boundary conditions, a_i can be regarded as a set of generalized coordinates. Substituting this into the energy function we obtain an algebraic function which can be treated using Poincare's general branching theory of finite dimensions.[8] Following Thompson[7] in writing the secondary path in the parametric form

$$a_i=a_i(a_1),\quad P=P(a_1) \tag{8}$$

and substituting into the equilibrium equation

$$(\partial W/\partial a_i)=0 \tag{9}$$

we can differentiate Eq. (9) with respect to a_1, generating as many perturbation equations as required. Continuing this procedure, which is a discrete version of the continuum perturbation method, the intitial post buckling for a system with a diagonal and totally symmetric energy function can be written in the following form

$$P=P^c+\frac{1}{2}\Lambda_{11}\Big|_c a_1^2 \tag{10}$$

where the initial curvature is

$$\Lambda_{11}\Big|_c=-\frac{A_{1111}}{3A'_{11}}\Big|_c$$

$$=-\frac{\partial^4 W_4}{\partial a_1^4}\frac{1}{3\left(\frac{\partial^2 W_2}{\partial a_1^2}\right)}\quad;\quad(\;)'=\frac{d(\;)}{dP} \tag{11}$$

and the perturbation parameter a_1 is the maximal buckling amplitude. Since our functional is diagonal and totally symmetric, the initial curvature is

$$\Lambda_{11}\Big|_c=D_x\left(\frac{i\pi}{\ell}\right)^2\frac{1}{4}\left(1-\frac{3\beta}{i^4}\right);\quad\beta=\frac{k_y}{D_x(\pi/\ell)^4r^2} \tag{12}$$

For the special case of a long isotropic cylindrical shell, the result is identical to that previously given.[1] Thus, for an orthotropic cylindrical shell both the stable symmetric and the unstable symmetric points of bifurcation can occur depending on the stiffness ratio β.

III. Extended Initial Post Buckling Analysis – Large Deflection

Now it is clear that we do not need to break our perturbation procedure immediately after the initial curvature (or slope in the case of an asymmetric point of bifurcation.[2] In fact, we could continue the perturbation process as far as we consider it necessary. Employing the Einstein summation convention and omitting the algebraic manipulations, the next secondary path derivative evaluated at the critical point is

$$\Lambda_{1111}\Big|_c=-\frac{1}{5A^*_{11}}\left[A_{111111}+10A_{s111}\left(\frac{-A_{s111}}{A_{ss}}\right)+10\overset{*}{A}_{1111}\Lambda_{11}\right]\Big|_c \tag{13}$$

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*Assistant Professor, presently Civil Eng. Department, Riyadh University, Saudi Arabia. Member AIAA.

where $(^*) = (d(\ )/d\Lambda)$, Λ is the loading parameter, c and s denote the critical and noncritical, respectively,

$$\frac{-A_{s111}}{A_{ss}}\bigg|_c = a_{s111}\bigg|_c$$

and a subscript denotes differentiation with respect to the corresponding generalized coordinates. Omitting the lengthy but simple calculations, the results can be given in the following closed-form

$$\bar{\Lambda}_{1111}\bigg|_c = \frac{\Lambda_{1111}}{D_x\left(\frac{\pi}{\ell}\right)^2}\bigg|_c = \frac{1}{64}\left[\frac{\beta(25/i^2) - \beta/66i^2 + 81i^6}{i^4 - \beta/9}\right] \quad (14)$$

For the special case of a long cylindrical shell, it is interesting to note that only an unstable symmetric point of bifurcation exists

$$\bar{\Lambda}_{1111}\bigg|_c = -\left(\frac{1}{4}k_y\frac{a^2}{r^2} + \frac{270}{128}\sqrt{\beta}\frac{1}{4!}a^4\right) \quad (15)$$

IV. Estimation of the Lower Stability Limit in the General Case of Asymmetric Buckling

It is by now evident from a previous work[1] that an unstable initial post buckling for a symmetrical buckling configuration is a strong indication of a severe imperfection sensitivity for the general case (asymmetrical buckling).[2] Thus, the preceding analysis suggests the existence of such a severe imperfection sensitivity for a wide range of the stiffness parameters β. Consequently, we can anticipate a drastic reduction in the buckling load due to the 'durchschlag' effect.[9]

An adequate analysis for this case has to take into account the nonlinear coupling between the different buckling modes. This is very complicated and will, therefore, be given in detail elsewhere. However, it might be very useful to consider an estimate for the lower stability limit (durchschlaglast) along the same lines as that given in Refs. 4 and 5. Such an estimate, although only of an approximative nature, might prove to be an indispensable guide in classifying experimental data. This would also provide us with an idea about the result which can be expected from an exact solution, usually numerical and very complicated.

Performing an isometric transformation of the energy functional

$$V \to \mathfrak{J} \quad (16)$$

and neglecting higher-order terms, it was shown in Ref. 9 that this leads to

$$\mathfrak{J} = \int_0^\ell \left[\frac{k_x b^5}{1440(1-\nu^2)r^2}\ddot{w}^2 + D_y\left(\frac{\pi}{b}\right)^3 w^2 + \frac{P}{2}\dot{w}^2\right]2\pi r dx \quad (17)$$

where D_y is the bending stiffness in the circumferential direction, and b is the wavelength of the isometric transformation (see Fig. 2). Inserting

$$w = \sum_i^\infty a_i \sin\frac{i\pi}{\ell}x \quad (18)$$

in Eq. (17) the smallest snapping pressure (durchschlag pressure) is easily obtained as

$$P_\ell^c = \left(\frac{k_x D_y \pi^3}{gor^2}\right)^{1/2} \quad (19)$$

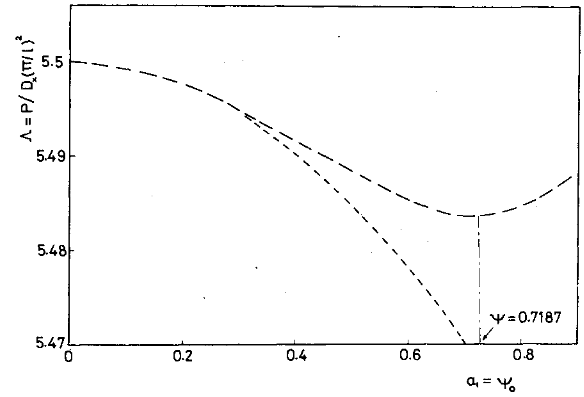

Fig. 1 Intitial and post buckling curves for $i=2$ and $\beta=6$, where Λ is the loading parameter, a_1 is the end rotation, ------ and – – – are the initial post buckling, and the extended initial post buckling curves respectively.

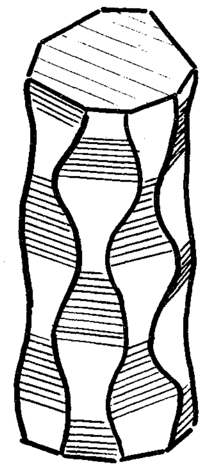

Fig. 2 Overall isometric transformation of a cylindrical surface under axial pressure.

For the special case of an isotropic cylindrical shell, we obtain

$$\sigma_\ell^c = 0.185\ Et/r \quad (20)$$

which is only very slightly different from that obtained by von Karman.[10-12]

V. Conclusion

The extension of Koiter's theory in the case of an axially compressed orthotropic cylindrical shell reflects not only a principle similarity to shell buckling behavior as that indicated by the initial post buckling approach (that is to say the initial post buckling is, for a wide range of parameters, unstable), but also a great similarity to the whole post buckling range[11,12] including the "von Karman" minimum buckling load (see Fig. 1). In this analysis the minimum load is, of course, only slightly smaller than the classical buckling load. However the important point is that it is an exact value obtained from an exact intrinsic formulation, and that it reflects the main characteristics of the large deflection post buckling.[11,12] The experimentally observed influence of stiffness and shell length on the stability performance of axially compressed cylinders are resembled in principle by Eqs. (12) and (15). Finally, the lower stability limit of the asymmetrical buckling of the orthotropic cylinder, obtained using an isometric energy functional is consistent with previous results concerning isotropic cylindrical shells.

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